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A stochastic model for the motion of two relativistic particles

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Abstract. Within the framework of the Kershaw stochastic model and of the hypothesis on the space stochasticity, equations of motion for two relativistic stochastic particles are obtained which coincide in form with the equations of Cufaro Petroni and Vigier. In the nonrelativistic limit, these equations of free motion reduce to the usual two-particle Schrödinger equation, the imaginary part of which corresponds to a Hamilton–Jacobi equation for two particles interacting through a nonlocal quantum potential.

In our recent papers (Namsrai 1980a, b, c) the motion of a single particle and of two interacting nonrelativistic particles has been investigated in terms of space stochasticity and we have obtained stochastic mechanics due to Nelson (1966), Kershaw (1964) and de la Pena and Cetto (1975). A method is proposed for the relativisation of the given scheme for describing the processes in the stochastic space; by using this method, the equations of a single-particle motion can be written in a covariant form which coincides with the equations of Lehr and Park (1977), Guerra and Ruggiero (1978) and Vigier (1979).

Now we consider the problem of two identical correlated relativistic scalar particles since this is the problem of real physical interest. We want to analyse this question within the framework of Kershaw's (1964) stochastic model and of the stochastic space $R_4(\hat{x}^\mu)$ with a small stochastic component (Namsrai 1980a) by using the Smoluchowski type equations for the probability density $\rho(x_1^\mu, x_2^\nu, s_1, s_2)$ of finding the first particle at point x_1^μ and the second one at x_2^ν at 'time' s_1 and s_2, respectively. Let $v_1^\mu(x_1^\lambda, x_2^\nu, s_1, s_2)$ and $v_2^\nu(x_1^\lambda, x_2^\mu, s_1, s_2)$ be their relative velocities. Here s is a certain invariant parameter (proper time), another interpretation of which can be found in Namsrai (1980a) and Miura (1979). We have assumed (Namsrai 1980a) that the derivative with respect to s could be interpreted as a derivative with respect to the direction of some (arbitrarily chosen) vector V^μ. Especially, if V^μ is the particle velocity, then s may be interpreted as the proper time of this particle.

A basic hypothesis for a generalisation of our scheme to the relativistic case was the following:

(i) The stochasticity of the space $R_4(\hat{x}^\mu)$ appears in the Euclidean region of the variables $\hat{x}_E^\mu = x_E^\mu + b_E^\mu$ only, x_E^μ being the regular part of the coordinate and b_E^μ some small random vector with a distribution $\lambda(b_E^\mu)$ obeying the conditions

$$\int d\lambda(b_E^\mu) = 1, \qquad d\lambda(b_E^\mu) \geq 0.$$

(ii) A shift of the coordinate $x^0 \to x^0 + i\tau$ is equivalent to the consideration of the physical quantities as functions of complex times $t + i\tau$ in the limit $\tau \to 0$, where τ is a random variable. The importance of this shift in the time variable has been noted by Alebastrov and Efimov (1974) and Davidson (1978) (see also Namsrai 1980d).

Our starting point is just the two-particle generalisation in the configuration space of our one-particle model (Namsrai 1980a, b, c). This model can be mathematically described in an eight-dimensional configuration space (see also Cufaro Petroni and Vigier (1979)) where a pair position and relative velocity are defined by the eight-component vectors X^i and v^i ($i = 1, \ldots, 8$) respectively, where

$$\{X^i\}_{i=1,\ldots,8} \equiv \{x_1^\mu, x_2^\nu\}_{\mu,\nu=0,\ldots,3}, \qquad \{v^i(X^i, s_1, s_2)\} \equiv \{v_1^\mu, v_2^\nu\}$$

with x_1^μ, x_2^ν four-vectors of the position of each body. The metric is defined by g_{ij} as in Cufaro Petroni and Vigier (1979) so that

$$X^2 = X_i X^i = g_{ij} X^i X^j = (x_1)^2 + (x_2)^2.$$

If $x_1^\mu(s_1), x_2^\nu(s_2)$ are the trajectories for the two particles, the trajectory in the configuration space will be $X^i(s_1, s_2)$.

By analogy with the three- and four-dimensional cases, we introduce here a two-particle version $\Psi(X^i, s_1, s_2, \Delta s_1, \Delta s_2)$ of the conditional probability densities $P(\boldsymbol{x}, t, \Delta t)$ and $\Psi(x^\mu, s, \Delta s)$ determined in Namsrai (1980a, b, c) and Lehr and Park (1977). If two particles do not correlate, the quantity of $\Psi(X^i, s_1, s_2, \Delta s_1, \Delta s_2)$ may be factorised by

$$\Psi(X^i, s_1, s_2, \Delta s_1, \Delta s_2) = \Psi_1(x_1^\mu, s_1, \Delta s_1)\Psi_2(x_2^\nu, s_2, \Delta s_2).$$

Here we will choose the gauge $\Delta s_1 = \Delta s_2 = \Delta s$.

Then Smoluchowski-type equations for $\rho(X^i, s_1, s_2)$ acquire the following form:

$$\begin{aligned} &\rho(X^i, s_1 \pm \Delta s, s_2 \pm \Delta s) \\ &\quad = \int d^8 Y_E\, \rho(X^I \mp Y^I, X^1 + iY_E^1, X^5 + iY_E^5; s_1, s_2) \\ &\qquad \times \Psi^\pm(X^I \mp Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2, \Delta s, Y_E^i) \end{aligned} \tag{1}$$

where X^I and Y^I ($I = 2, \ldots, 4, 6, \ldots, 8$) denote the space coordinates of X^i and Y^i.

By using the exact form of $\Psi^\pm(X^I \mp Y^I, X^1 + iY_E^1, \ldots, Y_E^i)$

$$\begin{aligned} &\Psi^\pm = (4\pi D_\pm \Delta s)^{-4} \exp[-(Y_E^i - Y_\pm^i)^2/4D_\pm \Delta s] \\ &Y_\pm^i = (\pm i v_\pm^1 \Delta s, \pm i v_\pm^5 \Delta s, v_\pm^I \Delta s) \end{aligned} \tag{2}$$

we obtain from (1) the following equations in the limit $\Delta s \Rightarrow 0$

$$\begin{aligned} &\partial\rho/\partial s_1 + \partial\rho/\partial s_2 + \partial_i(\rho v_+^i) - D_+ \Box\rho = 0 \\ &\partial\rho/\partial s_1 + \partial\rho/\partial s_2 + \partial_i(\rho v_-^i) + D_- \Box\rho = 0 \\ &\partial_i = \partial/\partial X^i, \qquad -\partial_i\partial^i = \Box = \Box_1 + \Box_2. \end{aligned} \tag{3a}$$

Here we assume $D_- = D_+ = D$, D is the diffusion coefficient, the quantities v_+^i and v_-^i are called the forward and backward velocity, respectively. We pass to the variables

$$v^i = \tfrac{1}{2}(v_+^i + v_-^i), \qquad u^i = \tfrac{1}{2}(v_+^i - v_-^i)$$

and sum (subtract) the equations in ($3a$) in pairs, thus obtaining

$$\begin{aligned} &\partial\rho/\partial s_1 + \partial\rho/\partial s_2 + \partial_i(\rho v^i) = 0 \\ &u^i = -D\,\partial^i \ln\rho \end{aligned} \tag{3b}$$

where $v^i(X^j, s_1, s_2)$ and $u^i(X^j, s_1, s_2)$ are the total drift and stochastic velocities.

In our model, as a generalisation of the assumption (Namsrai 1980a) that the mass (interpreted as the probability density times volume) cannot disappear through any hyperplanes characterised by the vectors v_1^μ and v_2^ν, respectively, we make the physical hypothesis that the total number of particles (i.e. pair in the real space–time) is conserved, and thus we write

$$\begin{aligned} \frac{\partial\rho}{\partial s_1} + \frac{\partial\rho}{\partial s_2} &= (\mathrm{d}\rho\cdot \boldsymbol{v}_1) + (\mathrm{d}\rho\cdot\boldsymbol{v}_2) \\ &= \frac{\partial\rho}{\partial x_1^0} v_1^0 + \frac{\boldsymbol{v}_1}{(1-\beta_1^2)^{1/2}}\frac{\partial\rho}{\partial\boldsymbol{x}_1} + \frac{\partial\rho}{\partial x_2^0} v_2^0 + \frac{\boldsymbol{v}_2}{(1-\beta_2^2)^{1/2}}\frac{\partial\rho}{\partial\boldsymbol{x}_2} \\ &= 0 \qquad (\beta_i^2 = \boldsymbol{v}_i^2/c^2,\ i = 1, 2) \end{aligned}$$

so that our continuity equation in the configuration space is

$$\partial_i(\rho v^i) = 0 \tag{4}$$

or in terms of v^i and u^i,

$$-u^i v_i + D\,\partial^i v_i = 0. \tag{5}$$

Due to Kershaw (1964) we can constitute the equations of the type (1) for the mean velocities $v_\pm^i(X^j, s_1, s_2)$ in the external fields $F_\pm^i(X^j, s_1, s_2)$ by the following formula

$$\begin{aligned} v_\pm^i(X^j, s_1+\varepsilon\Delta s, s_2+\varepsilon\Delta s) &= \frac{1}{N_\pm}\int\Big[v_\pm^i(X^I-\varepsilon Y^I, X^1+\mathrm{i}Y_{\mathrm{E}}^1, X^5+\mathrm{i}Y_{\mathrm{E}}^5, s_1, s_2) \\ &\quad + \varepsilon\frac{\Delta s}{M}F_\pm^i(X^I-\varepsilon Y^I, X^1+\mathrm{i}Y_{\mathrm{E}}^1, X^5+\mathrm{i}Y_{\mathrm{E}}^5, s_1, s_2)\Big] \\ &\quad \times\Psi^\pm(X^I-\varepsilon Y^I, X^1+\mathrm{i}Y_{\mathrm{E}}^1, X^5+\mathrm{i}Y_{\mathrm{E}}^5, s_1, s_2, \Delta s, Y_{\mathrm{E}}^i) \\ &\quad \times\rho(X^I-\varepsilon Y^I, X^1+\mathrm{i}Y_{\mathrm{E}}^1, X^5+\mathrm{i}Y_{\mathrm{E}}^5, s_1, s_2)\,\mathrm{d}^8Y_{\mathrm{E}} \end{aligned} \tag{6}$$

where

$$N_\pm = \int \mathrm{d}^8Y_{\mathrm{E}}\,\Psi^\pm(X^I-\varepsilon Y^I, X^1+\mathrm{i}Y_{\mathrm{E}}^1, \ldots, Y_{\mathrm{E}}^i)\rho(X^I-\varepsilon Y^I, \ldots, s_2)$$

are the normalisation constants and

$$\varepsilon = \begin{cases} 1 & \text{for } v_+^i \\ -1 & \text{for } v_-^i. \end{cases}$$

M is some effective mass (it can be of a matrix form with respect to m—mass of the scalar particle) of our two-body system.

In our case the equations (6) imply

$$\frac{\partial u_\pm^i}{\partial s_1} + \frac{\partial u_\pm^i}{\partial s_2} + u_\pm^i\,\partial_j u_\pm^i = F_\pm^i/M \pm D\Big(\frac{2}{D}u^j\,\partial_j u_\pm^i + \Box u_\pm^i\Big). \tag{7}$$

We sum equations (7) and have

$$D_c v^i - D_s u^i = (1/2M)(F^i_+ + F^i_-) = F^i/M$$
$$D_c = \frac{\partial}{\partial s_1} + \frac{\partial}{\partial s_2} + v^i \partial_i \qquad (8)$$
$$D_s = u^i \partial_i + D\Box.$$

Equation (8) together with the continuity equation (4) are the covariant analogy of the one-particle case in the two-particle system.

Let us notice that the left-hand side of equation (8) coincides exactly with the expression for acceleration obtained by Cufaro Petroni and Vigier (1979) on the basis of some assumption in the framework of the mathematical approach of Nelson (1967) and Guerra and Ruggiero (1978).

By equations (5) and (8) we have a coupled pair of nonlinear partial differential equations which may be linearised if we assume as before (Nelson 1966, 1967 and Cufaro Petroni and Vigier 1979)

$$v^i = \partial^i \varphi / m \qquad (9)$$

where $\varphi(X^i, s_1, s_2)$ is the phase function which is given by

$$\varphi(X^i, s_1, s_2) = \tfrac{1}{2} mc^2 (s_1 + s_2) + S(X^i). \qquad (10)$$

Starting from (4), (8) and using (3*b*), (9), (10) we obtain in the case $F^i \equiv 0$ an Hamilton–Jacobi-type equation ($R = \rho^{1/2}$, $D = \hbar/(2m)$) for our two-body system i.e.

$$(\partial_i \partial^i - \partial_i S \partial^i S/\hbar - 2m^2c^2/\hbar^2) R = 0 \qquad (11)$$

which yields for the continuity equation the form

$$2\partial_i R \partial^i S + R \partial_i \partial^i S = 0.$$

Finally, we have the equation for $\psi = R \exp(\mathrm{i}S/\hbar)$

$$(\Box - 2m^2c^2/\hbar^2)\psi = 0. \qquad (12)$$

In the nonrelativistic limit, the relation (12) reduces to the usual two-particle Schrödinger equation which (writing $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = R(\boldsymbol{x}_1, \boldsymbol{x}_2, t)\exp(\mathrm{i}S/\hbar)$) splits into the real and imaginary parts i.e.:

$$\partial P/\partial t + \boldsymbol{\nabla}_1 (P\, \boldsymbol{\nabla}_1 S/m) + \boldsymbol{\nabla}_2 (P\, \boldsymbol{\nabla}_2 S/m) = 0$$

with $P = R^2 = \psi^* \psi$ and

$$\partial S/\partial t + (\boldsymbol{\nabla}_1 S)^2/m + (\boldsymbol{\nabla}_2 S)^2/m + Q = 0$$

where

$$Q = -\hbar^2 (\nabla_1^2 R/R + \nabla_2^2 R/R)/2m$$

is a nonlocal quantum potential.

In conclusion we notice that the discussion of the physical implications of the above mentioned results have been found in Cufaro Petroni and Vigier (1979).

References

Alebastrov V A and Efimov G V 1974 *Commun. Math. Phys.* **38** 11–28
Cufaro Petroni N and Vigier J P 1979 *Lett. Nuovo Cimento* **26** 149–53
Davidson M 1978 *J. Math. Phys.* **19** 1975–8
Guerra F and Ruggiero P 1978 *Lett. Nuovo Cimento* **23** 529–34
Kershaw D 1964 *Phys. Rev.* **136**B 1850–6
Lehr W and Park J 1977 *J. Math. Phys.* **18** 1235–40
Miura T 1979 *Prog. Theor. Phys.* **61** 1521–35
Namsrai Kh 1980a *Foundations of Physics* **10** 353–61
—— 1980b *Foundations of Physics* **10** 731–42
—— 1980c *Commun. JINR* E2-80-56 (Dubna)
—— 1980d *Int. J. Theor. Phys.* **19** 397–404
Nelson E 1966 *Phys. Rev.* **150** 1079–85
—— 1967 *Dynamical Theories of Brownian Motion* (Princeton, NJ: University Press)
de la Pena L and Cetto A M 1975 *Foundations of Physics* **5** 355–70
Vigier J P 1979 *Lett. Nuovo Cimento* **24** 265–72